

$P - V$ criticality of higher dimensional black holes with nonlinear source

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In this paper, we consider the solutions of Einstein gravity in the presence of power Maxwell invariant source instead of linear electrodynamic. Following the method of Ref. [7], we investigate the analogy of nonlinear charged black hole solutions with the Van der Waals liquid–gas system.

I. INTRODUCTION

Theoretically one may be expected the cosmological constant term arises from vacuum expectation value of a quantum field and hence can vary. Therefore, it may be considered in the first law of thermodynamics with its conjugate [1–3]. By this generalization, the cosmological constant and its conjugate can be interpreted as geometrical pressure and volume of a black object system, respectively. Moreover, this approach leads to interesting conjecture on reverse isoperimetric inequality for black holes in contrast to Euclidean version of isoperimetric inequality. Regarding the inequality conjecture, some of the black hole processes may be restricted [4–6].

Furthermore, the extension of thermodynamic phase space has dramatic effects on the studying of famous phase transition of black holes in AdS space [6, 8, 9] and improves the analogy between small/large black hole with the Van der Waals liquid/gas phase transitions. Indeed, the AdS charged black holes exhibit an interesting phase transition with the same critical behavior as Van der Waals model, qualitatively [6].

In addition to Reissner–Nordström black holes, the effects of nonlinear electromagnetic field of static and rotating AdS black holes have been analyzed [7]. It has been shown that for the Born–Infeld black holes, one may obtain the same qualitative behavior (critical exponents) as in the Maxwell case.

In the last five years, it has introduced a class of nonlinear electrodynamics, the so called power

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Maxwell invariant (PMI) field (for more details, see [10]). The PMI field is significantly richer than that of the Maxwell field and in the special case ($s = 1$) it reduces to linear electromagnetic source. Its interesting black hole solutions with different thermodynamics and geometric properties have been examined before [10].

In this paper, we consider a spherically symmetric black hole solutions of Einstein gravity in the presence of PMI source. Regarding the cosmological constant as thermodynamic pressure, we investigate the analogy of black holes with Van der Waals liquid–gas system.

II. BLACK HOLES PHASE TRANSITION WITH PMI SOURCE

The bulk action of Einstein-PMI gravity has the following form

$$I_b = -\frac{1}{16\pi} \int_M d^{n+1}x \sqrt{-g} \left(R + \frac{n(n-1)}{l^2} + (-\mathcal{F})^s \right), \quad (1)$$

where $\mathcal{F} = F_{\mu\nu}F^{\mu\nu}$. We consider a spherically symmetric spacetime as

$$ds^2 = -V(r)dt^2 + \frac{dr^2}{V(r)} + r^2 d\Omega_{d-2}^2, \quad (2)$$

where $d\Omega_d^2$ stands for the standard element on S^d . Considering the field equations following from the variation of the bulk action with Eq. (2), one can show that the function $V(r)$ and gauge potential one-form are given by

$$V(r) = 1 + \frac{r^2}{l^2} - \frac{m}{r^{n-2}} + \frac{(2s-1)^2 \left(\frac{(n-1)(2s-n)^2 q^2}{(n-2)(2s-1)^2} \right)^s}{(n-1)(n-2s)r^{2(ns-3s+1)/(2s-1)}}, \quad (3)$$

$$A = -\sqrt{\frac{n-1}{2(n-2)}} q r^{(2s-n)/(2s-1)} dt, \quad (4)$$

$$F = dA.$$

The power $s \neq n/2$ denotes the nonlinearity parameter of the source which is restricted to $s > 1/2$ [10], and the parameters m and q are related to the ADM mass M and the electric charge Q of the black hole

$$M = \frac{\omega_{n-1}}{16\pi} (n-1)m, \quad (5)$$

$$Q = \frac{\sqrt{2}(2s-1)s \omega_{n-1}}{8\pi} \left(\frac{n-1}{n-2} \right)^{s-1/2} \left(\frac{(n-2s)q}{2s-1} \right)^{2s-1}, \quad (6)$$

where ω_{n-1} is given by

$$\omega_{n-1} = \frac{2\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)}. \quad (7)$$

There is an essential singularity at $r = 0$ and therefore this solution may interpreted as black hole. Using the surface gravity relation, we can obtain the temperature of the black hole solutions as

$$T = \frac{V'(r_+)}{4\pi} = \frac{n-2}{4\pi r_+} \left(1 + \frac{n}{n-2} \frac{r_+^2}{l^2} - \frac{(2s-1) \left(\frac{(n-1)(2s-n)^2 q^2}{(n-2)(2s-1)^2} \right)^s}{(n-1)(n-2)r_+^{2(ns-3s+1)/(2s-1)}} \right), \quad (8)$$

with r_+ denotes the radius of the event horizon which is the largest root of $V(r_+) = 0$. The electric potential Φ , measured at infinity with respect to the horizon and the black hole entropy S , determined from the area law may be written as

$$\Phi = \sqrt{\frac{n-1}{2(n-2)}} \frac{q}{r_+^{(n-2s)/(2s-1)}}, \quad (9)$$

$$S = \frac{\omega_{n-1} r_+^{n-1}}{4}. \quad (10)$$

Now, as it has considered before [6], we interpret Λ as a thermodynamic pressure P ,

$$P = -\frac{1}{8\pi} \Lambda = \frac{n(n-1)}{16\pi l^2}, \quad (11)$$

where its corresponding conjugate quantity is the thermodynamic volume [5]

$$V = \frac{\omega_{n-1} r_+^n}{n}. \quad (12)$$

Considering obtained quantities, one can show that they satisfy the following Smarr formula

$$M = \frac{n-1}{n-2} TS + \frac{ns-3s+1}{s(2s-1)(n-2)} \Phi Q - \frac{2}{n-2} VP. \quad (13)$$

It has been shown that Eq. (13) may be obtained by a scaling dimensional argument [4, 11]. In addition, the (extended phase-space) first law of thermodynamics cab be written as

$$dM = TdS + \Phi dQ + VdP. \quad (14)$$

In what follows we shall study the analogy of the liquid–gas phase transition of the Van der Waals fluid with the phase transition in black hole solutions in the presence of PMI source.

A. Equation of state

Using the Eqs. (11) and (8) for a fixed charge Q , one may obtain the equation of state, $P(V, T)$

$$P = \frac{(n-1)}{4r_+} T - \frac{(n-1)(n-2)}{16\pi r_+^2} + \frac{1}{16\pi} \frac{(2s-1) \left(\frac{(n-1)(2s-n)^2 q^2}{(n-2)(2s-1)^2} \right)^s}{r_+^{2s(n-1)/(2s-1)}}, \quad (15)$$

where r_+ is a function of the thermodynamic volume, V (see Eq. (12)). Following [6], we identify the geometric quantities P and T with physical pressure and temperature of system by using dimensional analysis and $l_P^{n-1} = G_{n+1}\hbar/c^3$ as

$$[\text{Press}] = \frac{\hbar c}{l_P^{n-1}}[P], \quad [\text{Temp}] = \frac{\hbar c}{k}[T]. \quad (16)$$

Therefore the physical pressure and physical temperature are given by

$$\begin{aligned} \text{Press} &= \frac{\hbar c}{l_P^{n-1}}P = \frac{\hbar c}{l_P^{n-1}} \frac{(n-1)T}{4r_+} + \dots \\ &= \frac{k\text{Temp}(n-1)}{4l_P^{n-1}r_+} + \dots \end{aligned} \quad (17)$$

Now, one could compare them with the Van der Waals equation [6], and identify the specific volume v of the fluid with the horizon radius as $v = \frac{4r_+l_P^{n-1}}{n-1}$, and in geometric units ($l_P = 1$, $r_+ = (n-1)v/4$), one can write the equation of state (15) in the following form

$$P = \frac{T}{v} - \frac{(n-2)}{\pi(n-1)v^2} + \frac{1}{16\pi} \frac{\kappa q^{2s}}{v^{2s(n-1)/(2s-1)}}, \quad (18)$$

$$\kappa = \frac{4^{2s(n-1)/(2s-1)}(2s-1) \left(\frac{(n-1)(2s-n)^2}{(n-2)(2s-1)^2} \right)^s}{(n-1)^{2s(n-1)/(2s-1)}}, \quad (19)$$

Considering Eq. (18), we plot the $P - V$ isotherm diagram in Figs. 1 and 2. These figures show that, similar to Van der Waals gas, there is a critical point which is a point of inflection on the critical isotherm. The pressure and volume at the critical point are known as the critical pressure and the critical volume, respectively. Above the critical point and for large volumes and low pressures, the isotherms lose their inflection points and approach equilateral hyperbolas, the so-called, the isotherms of an ideal gas. It is shown that the slope of the isotherm $P - V$ diagram passing through the critical point is zero. Furthermore, as we mentioned before, the critical point is a point of inflection on the critical isotherm, hence

$$\frac{\partial P}{\partial v} = 0, \quad (20)$$

$$\frac{\partial^2 P}{\partial v^2} = 0. \quad (21)$$

Using the Eqs. (20) and (21) with the equation of state, we will be able to calculate the critical parameters

$$v_c = \left[\frac{\kappa s(n-1)^2(2ns-4s+1)q^{2s}}{16(n-2)(2s-1)^2} \right]^{(2s-1)/[2(ns-3s+1)]}, \quad (22)$$

$$T_c = \frac{4(n-2)(ns-3s+1) \left[\frac{\kappa s(n-1)^2(2ns-4s+1)q^{2s}}{16(n-2)(2s-1)^2} \right]^{(1-2s)/[2(ns-3s+1)]}}{\pi(n-1)(2ns-4s+1)}, \quad (23)$$

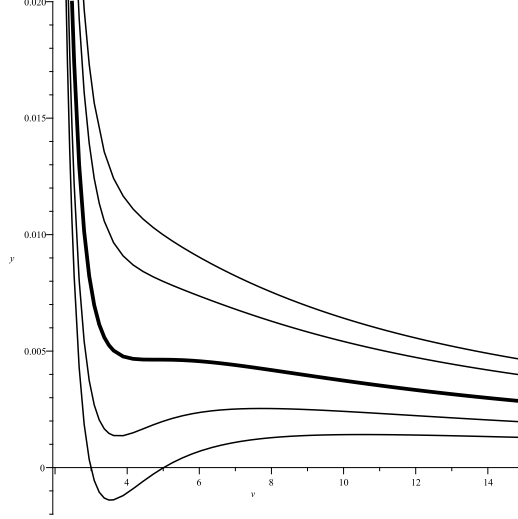


FIG. 1: $P - V$ diagram of charged AdS black hole in PMI for $s = \frac{3}{4}$ and $n = 3$. The temperature of isotherms decreases from top to bottom. Bold line is the critical isotherm diagram.

$$P_c = \frac{(n-2)(ns-3s+1)}{\pi s(n-1)^2 \left[\frac{\kappa s(n-1)^2(2ns-4s+1)q^{2s}}{16(n-2)(2s-1)^2} \right]^{(2s-1)/(ns-3s+1)}}. \quad (24)$$

These relations lead us to obtain the following universal ratio

$$\rho_c = \frac{P_c v_c}{T_c} = \frac{2ns-4s+1}{4s(n-1)}. \quad (25)$$

Note that for $s = -2/(n-5)$ with arbitrary spacetime dimensions, one can recover the ratio $\rho_c = 3/8$, characteristic for a Van der Waals gas.

B. Gibbs Free energy

The thermodynamic behavior of a system governed by the thermodynamic potentials such as the Gibbs free energy. It is known that the Gibbs free energy of a gravitational system may be obtained by evaluating the Euclidean on-shell action. In order to calculate it, we use the counterterm method for canceling of divergences. Furthermore, to make action well-defined, one should add the Gibbons-Hawking boundary term to the bulk action. In addition, to fix charge on the boundary (working in canonical ensemble) we should consider a boundary term for electromagnetic field. So the total action is

$$I = I_b + I_{ct} - \frac{1}{8\pi} \int_{\partial M} d^n x \sqrt{\gamma} K - \frac{s}{4\pi} \int_{\partial M} d^n x \sqrt{\gamma} (-\mathcal{F})^{s-1} n_\mu F^{\mu\nu} A_\nu, \quad (26)$$

where I_{ct} is the counterterm action, and γ_{ij} and K denote the induce metric and extrinsic curvature of the boundary. Using Eq. (26), it is straightforward to calculate the on-shell value of the total

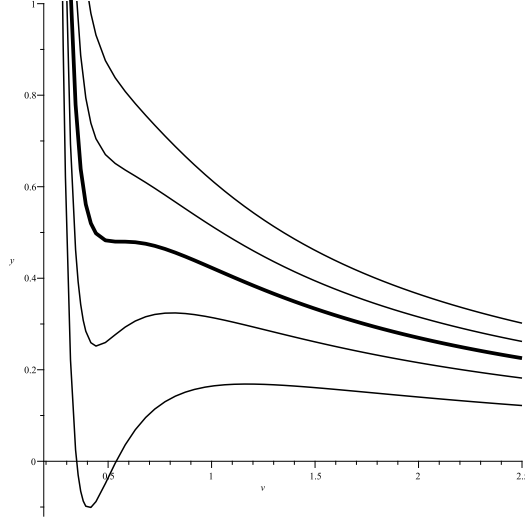


FIG. 2: $P-V$ diagram of charged AdS black hole in PMI for $s = 2$ and $n = 5$. The temperature of isotherms decreases from top to bottom. Bold line is the critical isotherm diagram.

action

$$I = \frac{\beta\omega_{n-1}}{16\pi} \left[\frac{l^2 r_+^{n-2} - r_+^n}{l^2} + \frac{(2s-1)(2sn-4s+1)\Psi^s}{(n-1)(n-2s)} r_+^n \right], \quad (27)$$

where

$$\Psi = \left(\frac{n-1}{n-2} \right) \left(\frac{2s-n}{2s-1} \right)^2 q^2 r_+^{-\frac{2(n-1)}{2s-1}}.$$

and β is the periodic Euclidean time which is related to inverse of Hawking temperature. Using the fact that $G = I\beta^{-1}$ with Eq.(11), the (fixed charge) Gibbs free energy in the extended phase space may be written as

$$G(T, P) = \frac{\beta\omega_{n-1}}{16\pi} \left[r_+^{n-2} - \frac{16\pi P}{n(n-1)} r_+^n + \frac{(2s-1)(2sn-4s+1)\Psi^s}{(n-1)(n-2s)} r_+^n \right]. \quad (28)$$

Using the fact that the Gibbs free energy, temperature and the pressure of the system are constant during the phase transition, one can plot coexistence curve of two phases (see Fig. 5).

C. Critical exponents

One of the most important characters of the phase transition is the value of its critical exponents. So, following the approach of [7], we calculate the critical exponents α , β , γ , δ for the phase transition of $(n+1)$ -dimensional charged black holes with arbitrary s . In order to obtain the

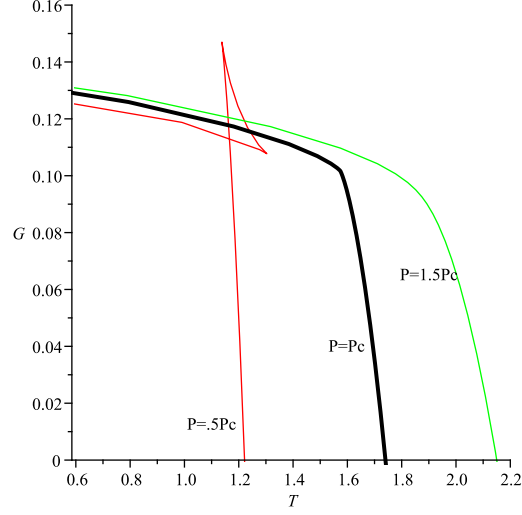


FIG. 3: Gibbs free energy density ($\frac{G(T,P)}{\omega_{n-1}}$) of charged black hole with PMI source with respect to temperature for $q = 1$, $s = \frac{6}{5}$ and $n = 4$.

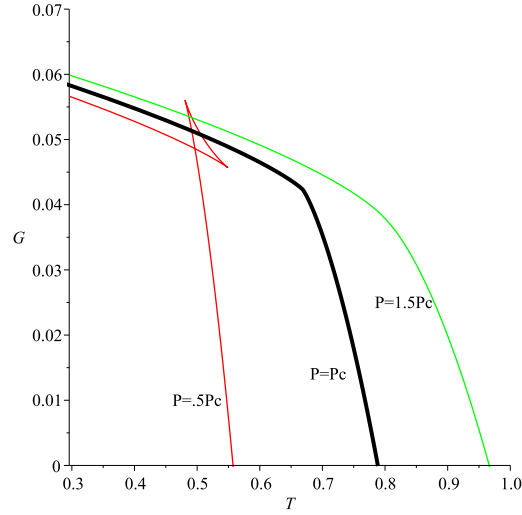


FIG. 4: Gibbs free energy density ($\frac{G(T,P)}{\omega_{n-1}}$) of charged black hole with PMI source with respect to temperature for $q = 1$, $s = \frac{3}{4}$ and $n = 3$.

critical exponent α we consider the entropy of horizon S and rewrite it in terms of T and V . So we have

$$S = S(T, V) = \left[\omega_{n-1} (nV)^{n-1} \right]^{\frac{1}{n}}. \quad (29)$$

Obviously, this is independent of T and then the specific heat vanishes, ($C_V = 0$), and hence $\alpha = 0$.

To obtain other exponents, we study equation of state (18) in terms of reduced thermodynamics

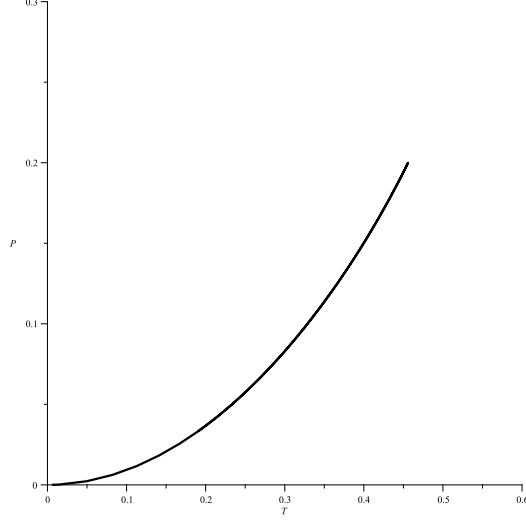


FIG. 5: Coexistence curve of charged black hole with PMI source for $q = 1$, $s = 2$ and $n = 7$.

variables

$$p = \frac{P}{P_c}, \quad \nu = \frac{v}{v_c}, \quad \tau = \frac{T}{T_c}. \quad (30)$$

So, (18) translates into the following reduced equation of state

$$p = \frac{4(n-1)s\tau}{(2ns-4s+1)\nu} - \frac{n-1}{(ns-3s+1)\nu^2} + \frac{(2s-1)^2}{(2ns-4s+1)(ns-3s+1)\nu^{\frac{2s(n-1)}{2s-1}}}. \quad (31)$$

To study the recent equation, we will slightly generalized the argument of [7] for nonlinear Maxwell theory. Indeed, we can rewrite the equation of state (31) as

$$p = \frac{1}{\rho_c} \frac{\tau}{\nu} + f(\nu, s), \quad (32)$$

where ρ_c stand for the critical ratio and

$$f(\nu, s) = \frac{1}{s(1-4\rho_c)} \left(\frac{1}{\nu^2} - \frac{\left(\frac{2s-1}{n-1}\right)^2}{4s\rho_c\nu^{\frac{2s(n-1)}{2s-1}}} \right).$$

The function $f(v, s)$ depends on v and s compared to [7] where it is independent of s . But as we will see parameter s does not play any dramatic role and does not change critical exponents. Following the method of Ref. [7], one may define two new parameters t and ω

$$\tau = t + 1, \quad \nu = (\omega + 1)^{1/\epsilon}, \quad (33)$$

where ϵ is a positive parameter. Now we can expand (31) near the critical point to obtain

$$p = 1 + At - Bt\omega - C\omega^3 + O(t\omega^2, \omega^4), \quad (34)$$

with

$$A = \frac{1}{\rho_c}, \quad B = \frac{1}{\epsilon \rho_c}, \quad C = \frac{2s(n-1)}{3\epsilon^2(2s-1)}. \quad (35)$$

We consider a fixed $t < 0$ and differentiate the Eq. (34) to obtain

$$dP = -P_c(Bt + 3C\omega^2)d\omega. \quad (36)$$

Now, we denote the volume of small and large black holes with ω_s and ω_l , respectively and apply the Maxwell's equal area law. One obtains

$$\begin{aligned} p &= 1 + At - Bt\omega_l - C\omega_l^3 = 1 + At - Bt\omega_s - C\omega_s^3 \\ 0 &= \int_{\omega_l}^{\omega_s} \omega dP. \end{aligned} \quad (37)$$

These equations leads to a unique non-trivial solution

$$\omega_s = -\omega_l = \sqrt{\frac{-Bt}{C}}, \quad (38)$$

and therefore we can find

$$\eta = V_c(\omega_l - \omega_s) = 2V_c\omega_l \propto \sqrt{-t} \quad \Rightarrow \quad \beta = \frac{1}{2}. \quad (39)$$

Now, we should calculate the next exponent, γ . In order to obtain it, one should consider Eq. (36)

$$\kappa_T = -\frac{1}{V} \frac{\partial V}{\partial P} \Big|_T \propto \frac{1}{P_c} \frac{1}{Bt} \quad \Rightarrow \quad \gamma = 1. \quad (40)$$

Next we calculate the final exponent, δ . To do this, we should obtain the shape of the critical isotherm $t = 0$ (34), i.e.,

$$p - 1 = -C\omega^3 \quad \Rightarrow \quad \delta = 3. \quad (41)$$

We conclude that the thermodynamic exponents associated with the nonlinear charged black hole in any dimension $n \geq 3$ with arbitrary nonlinearity parameter, s , coincide with those of the Van der Waals fluid (the same critical exponents as in the linear Maxwell case).

III. SUMMARY

In this paper, we have considered the cosmological constant and its conjugate quantity as thermodynamic variables and investigated the thermodynamic properties of a class of black hole

solutions. At the first step we have introduced the black hole solutions of the Einstein gravity- Λ in the presence of PMI source.

Then, we have used the Hawking temperature as an equation of state and calculated the critical parameters, T_C , v_C and P_C . We have plotted the isotherm diagram (P - V) of charged black holes in PMI theory and found that the total behavior of this diagram is the same as that of the Van der Waals gas. Also, we have obtained the Gibbs free energy of a gravitational system through the use of Euclidean on-shell action to investigate the thermodynamics behavior of the system.

Furthermore, we have calculated the critical exponent of the phase transition and concluded that the thermodynamic exponents associated with the nonlinear charged black hole in arbitrary dimension coincide with those of the Van der Waals fluid and indeed the mean field theory.

Finally, it is interesting to work in grand canonical ensemble which the potential, instead of charge, should be fixed on the boundary. In contrast to Maxwell case, here we see a phase transition in the grand canonical ensemble [12]. In addition, we are working on the existence of the phase transition in the BTZ like black holes [12].

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